Q. prove that the series $\sum_{n=1}^{\infty} \frac{3^n + 5^n}{15^n}$ Converges and find the sum.

Solution :-

$$\sum_{n=1}^{\infty} \frac{3^n+5^n}{15^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{15^n} + \frac{5^n}{15^n} \right)$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{\cancel{3}^{1}}{\cancel{15}_{5}} \right)^n + \left(\frac{\cancel{5}^{1}}{\cancel{15}_{3}} \right)^n \right)$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{1}{5} \right)^n + \left(\frac{1}{3} \right)^n \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{5} \right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n$$

now, $\sum_{n=1}^{\infty} \left(\frac{1}{5} \right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{5} \right) \left(\frac{1}{5} \right)^{n-1}$

The above relation is in a geometric series having

$a = \frac{1}{5}$ and $r = \frac{1}{2}$

Since $-1 < r < 1$, the condition of sum to be converges,

here, $r < 1$, $n \to \infty$

* Therefore, the sum is converges.

now the sum of the geometric series;

$\sum \left(\frac{1}{5} \right)^n$ is given by,

$$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n} \text{ is given by,}$$

$$S_{\infty} = \frac{a}{(1-r)} \quad (\text{when } n \to \infty)$$

$$= \frac{1/5}{\left(1 - \frac{1}{5}\right)}$$

$$= \frac{1/5}{\left(\frac{5-1}{5}\right)}$$

$$= \frac{1/5}{4/5}$$

$$= \frac{1}{\cancel{5}} \times \frac{\cancel{5}}{4}$$

$$\boxed{S_{\infty} = \frac{1}{4}}$$

Similarly, now,

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1}$$

$$a = \frac{1}{3}, \quad r = \frac{1}{3}, \quad n \to \infty$$

Since, $r < 1$, the given series is converges.

* Therefore, the series $\sum_{n=1}^{\infty} \frac{3^n + 5^n}{15^n}$, Converges.

now, the sum of the geometric series,

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n} \text{ is,}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n \text{ is,}$$

$$S_\infty = \frac{a}{(1-r)}$$

$$= \frac{1/3}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{1/3}{\left(\frac{3-1}{3}\right)}$$

$$= \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{2}$$

$$= \frac{1}{2}$$

Therefore, $\boxed{S_\infty = \frac{1}{2}}$ ✓

now, the total sum of the given series

$$\sum_{n=1}^{\infty}\frac{3^n+5^n}{15^n} = \sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$$

$$= \frac{1}{4} + \frac{1}{2}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{3^n+5^n}{15^n} = \frac{1+2}{4} = \frac{3}{4}}$$ ✓